

Relative category and monoidal topological complexity

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Abstract

If a map f has a homotopy retraction, then Doeraene and El Haouari conjectured that the sectional category and the relative category of f are the same. In this work we discuss this conjecture for some lower bounds of these invariants. In particular, when we consider the diagonal map, we obtain results supporting Iwase-Sakai's conjecture which asserts that the topological complexity is the monoidal topological complexity.

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Introduction.

The topological complexity, which can be defined as the sectional category of the diagonal map, is a numerical homotopy invariant introduced by Farber in [7] for the study of the motion planning problem in robotics. Iwase and Sakai ([12], [13]) introduced a relative version, called monoidal topological complexity, and conjectured that both notions are the same. First results supporting Iwase-Sakai's conjecture were given by Dranishnikov [6] but it still remains as an open problem.

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On the other hand, Doeraene and El Haouari introduced an approximation of the sectional category called *relative category* and proved that the difference between these two invariants is at most one. Then they ask in [5] for which cases the sectional category and the relative category agree. In such cases it would give important information about the map under consideration. As they have checked, in general these two invariants do not agree. However Doeraene and El Haouari conjectured that the equality holds as long as the map has a homotopy retraction.

In this paper we first show that Iwase-Sakai's conjecture is actually included in Doeraene-El Haouari's conjecture. Then we establish analogues versions of this conjecture for several approximations of both the sectional and relative categories.

1 Preliminary notions and results.

1.1 Sectional category and topological complexity.

The *sectional category* of a map $f : Y \rightarrow X$, $\text{secat}(f)$, is the least integer n (or ∞) such that X can be covered by $n + 1$ open subsets, over each of which f admits a homotopy section. When f is a fibration, then we can take local strict sections, recovering the usual notion of sectional category, or Schwarz genus [18], for fibrations. The *topological complexity*, $\text{TC}(X)$, of a space X in the sense of Farber [7] is the sectional category of the path fibration $\pi : X^I \rightarrow X \times X$, $\alpha \mapsto (\alpha(0), \alpha(1))$. Here X^I denotes the function space of all paths $\alpha : I \rightarrow X$, provided with the compact-open topology. Another important particular case is $\text{cat}(X)$, the *Lusternik-Schnirelmann category* of a (pointed) space X . If $PX = \{\alpha \in X^I : \alpha(0) = *\}$ is the path space of X and $ev : PX \rightarrow X$, $\alpha \mapsto \alpha(1)$ denotes corresponding fibration, evaluation at 1, then one has that $\text{cat}(X) = \text{secat}(ev)$.

The sectional category, which is a homotopy numerical invariant, can be characterized through the iterated join of $f : Y \rightarrow X$. Recall that, given any pair of maps $A \xrightarrow{\alpha} C \xleftarrow{\beta} B$, the *join of α and β* , $A *_C B \rightarrow C$, is obtained by taking the homotopy pushout of the homotopy pullback of α and β ,

Setting $j_f^0 = f : Y \rightarrow X$ and $*_X^0 Y = Y$ we can define inductively $j_f^n : *_X^n Y \rightarrow X$ as the join of $j_f^{n-1} : *_X^{n-1} Y \rightarrow X$ and $f : Y \rightarrow X$. We point out that here we are using $*^n$ to denote the join of $n + 1$ copies of the considered object. The characterization of sectional category is then given by the following classical result, see for instance [14] or [18].

Theorem 1. Let $f : Y \rightarrow X$ be a map. If X is a paracompact space, then one has $\text{secat}(f) \leq n$ if and only if $j_f^n : *_X^n Y \rightarrow X$ admits a homotopy section.

This theorem was first proved by Schwarz for a fibration $p : E \rightarrow B$, in which case the join j_p^n can be constructed to be a fibration and we may require the fibration j_p^n to have a strict section instead of a homotopy section. Indeed, if $\alpha : A \rightarrow C$ and $\beta : B \rightarrow C$ are fibrations, the join map $\alpha *_C \beta : A *_C B \rightarrow C$ can be explicitly described as follows $A *_C B = A \amalg (A \times_C B \times [0, 1]) \amalg B / \sim \rightarrow C$, $\langle a, b, t \rangle \mapsto \alpha(a) = \beta(b)$ where \sim is given by $(a, b, t) \sim a$ if $t = 0$ and $(a, b, t) \sim b$ if $t = 1$. This map is a fibration whose fibre is the ordinary join of the fibres.

Remark 2. In order to avoid the unnecessary technical requirement on the space X of being paracompact, we will consider the statement in Theorem 1 as the definition of sectional category of a map. In particular, if X is any topological space, then taking $f = \pi : X^I \rightarrow X \times X$ we have

$$\text{TC}(X) \leq n \iff j_\pi^n : *_X^n X^I \rightarrow X \times X \text{ admits a (homotopy) section.}$$

1.2 Relative category.

By construction of the iterated join $j_f^n : *_X^n Y \rightarrow X$ we obtain, for each $n \geq 0$, a homotopy commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\iota_n} & *_X^n Y \\ & \searrow f & \downarrow j_f^n \\ & & X. \end{array} \quad (1)$$

In particular, for $n = 0$, the map $\iota_0 : Y \rightarrow Y$ is just the identity.

Definition 3. [4] Let $f : Y \rightarrow X$ be a map. The *relative category* of f , denoted by $\text{relcat}(f)$, is the least integer n such that j_f^n admits a homotopy section σ which satisfies $\sigma f \simeq \iota_n$.

It is clear that $\text{secat}(f) \leq \text{relcat}(f)$. Doeraene and El Haouari proved in [4] that the difference between the two invariants is at most 1:

Theorem 4. For any map $f : Y \rightarrow X$ one has $\text{secat}(f) \leq \text{relcat}(f) \leq \text{secat}(f) + 1$.

They also set in [5] the following conjecture, that we will refer to as the *D-EH conjecture*:

Conjecture 1. (D-EH Conjecture) Let $f : Y \rightarrow X$ be any map. If $f : Y \rightarrow X$ admits a homotopy retraction, then $\text{secat}(f) = \text{relcat}(f)$.

Remark 5. The hypothesis of the existence of a homotopy retraction cannot be relaxed since, as pointed out in [4], for the Hopf map $f : S^3 \rightarrow S^2$, we have $\text{secat}(f) = 1$ while $\text{relcat}(f) = 2$. Another example is the inclusion $f : S^1 \hookrightarrow D^2$, for which $\text{secat}(f) = 0$ and $\text{relcat}(f) = 1$ (observe that, in general, $\text{relcat}(f) = 0$ if and only if f is a homotopy equivalence).

1.3 Monoidal topological complexity.

If X is a topological space, we denote by $\Delta : X \rightarrow X \times X$, $x \mapsto (x, x)$, the diagonal map and by $s_0 : X \rightarrow X^I$ the homotopy equivalence that associates with $x \in X$ the constant path \hat{x} in x , then we obviously have $\pi s_0 = \Delta$.

An important variant of the topological complexity is the *monoidal topological complexity*, which was introduced by Iwase and Sakai in [12].

Definition 6. [12] The *monoidal topological complexity* of X , $\text{TC}^M(X)$, is the least integer n such that $X \times X$ can be covered by $n + 1$ open sets $U_i \supseteq \Delta(X)$ over each of which there exists a section s_i of $\pi : X^I \rightarrow X \times X$ which satisfies $s_i \Delta = s_0$.

Again it is clear that $\text{TC}(X) \leq \text{TC}^M(X)$ and Iwase-Sakai proved that the difference between the two numbers is at most 1:

Theorem 7. [13] For any locally finite simplicial complex (or more generally, any Euclidean Neighborhood Retract) X , one has $\text{TC}(X) \leq \text{TC}^M(X) \leq \text{TC}(X) + 1$.

Iwase and Sakai also conjectured in [13] that the monoidal topological complexity coincides with the classical topological complexity. Their conjecture will be referred to as the *I-S conjecture*

Conjecture 2. (I-S Conjecture) For any locally finite simplicial complex X , one has $\text{TC}(X) = \text{TC}^M(X)$.

In [6], A. Dranishnikov shows that the equality holds under certain restrictions on the space X :

Theorem 8. [6] If X is a space, then the equality $\text{TC}(X) = \text{TC}^M(X)$ holds in the following cases:

- (i) X is a $(q - 1)$ -connected simplicial complex and $\dim(X) \leq q(\text{TC}(X) + 1) - 2$;
- (ii) X is a connected Lie group.

2 Monoidal topological complexity is a relative category.

Part (i) of previous theorem is based on the following characterization of TC^M (see Theorem 9). Using the explicit description of the join for fibrations we can see that Diagram (1) can be constructed in a commutative way. That is, for each $n \geq 0$, there exists a commutative diagram:

$$\begin{array}{ccc} X^I & \xrightarrow{\iota_n} & *_n^{X \times X} X^I \\ & \searrow \pi & \downarrow j_\pi^n \\ & & X \times X. \end{array}$$

As the map $s_0 : X \rightarrow X^I$, $x \mapsto \hat{x}$, satisfies $\pi s_0 = \Delta$, we set $s_n := \iota_n s_0$ and we have for any $n \geq 0$, $j_\pi^n s_n = \Delta$.

Theorem 9. [6] If X is paracompact, then $\text{TC}^M(X) \leq n$ if and only if the fibration $j_\pi^n : *_{X \times X}^n X^I \rightarrow X \times X$ admits a strict section σ such that $\sigma \Delta = s_n$.

This gives a characterization of TC^M which is very similar to the definition of relcat . Indeed, with this notation, $\text{relcat}(\Delta) \leq n$ if and only if $j_\pi^n : *_{X \times X}^n X^I \rightarrow X \times X$ admits a homotopy section σ such that $\sigma \Delta \simeq s_n$.

Remark 10. Again, as in the case of sectional category (and in particular for topological complexity) we will consider the statement of Theorem 9 as the definition of $\text{TC}^M(X)$ without requiring the space X to be paracompact.

We will prove that, under a non very restrictive condition on X , the equality $\text{TC}^M(X) = \text{relcat}(\Delta)$ holds. In order to see this, we use the following lemma, proved by Harper [11].

Lemma 11. Consider the diagram $X \xrightarrow{u} B \xleftarrow{\pi} E$, where π is a fibration with a strict section $\sigma : B \rightarrow E$. Suppose $\hat{u} : X \rightarrow E$ is a lift of u , that is, \hat{u} satisfies $\pi \hat{u} = u$. If σu is homotopic to \hat{u} , then σu is fibrewise homotopic to \hat{u} (over B).

Recall that a *locally equiconnected space* is a space X in which the diagonal map $\Delta : X \rightarrow X \times X$ is a (closed) cofibration. The class of locally equiconnected spaces is large enough. For instance, CW-complexes and metrizable spaces fit on such class.

Theorem 12. If X is a locally equiconnected space, then $\text{TC}^M(X) = \text{relcat}(\Delta)$.

Proof. Obviously, $\text{relcat}(\Delta) \leq \text{TC}^M(X)$. Now assume $\text{relcat}(\Delta) = n$ and consider $\sigma : X \times X \rightarrow *_{X \times X}^n X^I$ such that $j_\pi^n \sigma = \text{id}$ and $\sigma \Delta \simeq s_n$. Therefore, by previous lemma, we obtain $F : \sigma \Delta \simeq_{X \times X} s_n$ a fibrewise homotopy over $X \times X$. Now, as $(X \times X, \Delta(X))$ is a closed cofibred pair and j_π^n a fibration we can take a lift in the diagram

$$\begin{array}{ccc} X \times X \times \{0\} \cup \Delta(X) \times I & \xrightarrow{h} & *_{X \times X}^n X^I \\ \downarrow & \nearrow \tilde{h} & \downarrow j_\pi^n \\ X \times X \times I & \xrightarrow{pr} & X \times X, \end{array}$$

where h is the map defined as $h(x, y, 0) = \sigma(x, y)$ and $h(x, x, t) = F(x, t)$. Then, defining $\sigma' := \tilde{h} i_1$ we have that $j_\pi^n \sigma' = \text{id}$ and $\sigma' \Delta = s_n$. This means that $\text{TC}^M(X) \leq n$. \square

Corollary 13. The D-EH conjecture contains the I-S conjecture.

Proof. The diagonal map $\Delta : X \rightarrow X \times X$ admits the projection $p_2 : X \times X \rightarrow X$ as an obvious (homotopy) retraction. \square

Using this result we obtain a slight improvement of Theorem 8 part (ii).

Corollary 14. Let X be a connected CW H-space. Then

$$\mathrm{TC}(X) = \mathrm{TC}^M(X) = \mathrm{cat}(X) = \mathrm{cat}(X \times X / \Delta(X)).$$

Proof. It follows directly from Theorem 11 in [5]. See also [16] and [10]. \square

3 A stable version of D-EH conjecture.

In this section we prove that the D-EH conjecture holds after suspension. In order to make precise our statement we introduce approximations of the sectional category and relative category of a map in the same spirit as the σ^i -category (see [19] or [2]).

Let $i \geq 1$ be an integer and $f : Y \rightarrow X$ a map. By suspending i times Diagram (1) we get a homotopy commutative diagram:

$$\begin{array}{ccc} \Sigma^i Y & \xrightarrow{\Sigma^i \iota_n} & \Sigma^i *_X^n Y \\ & \searrow \Sigma^i f & \downarrow \Sigma^i j_f^n \\ & & \Sigma^i X. \end{array}$$

We then define:

- $\sigma^i \mathrm{secat}(f)$ to be the least integer n such that $\Sigma^i j_f^n$ admits a homotopy section;
- $\sigma^i \mathrm{relcat}(f)$ to be the least integer n such that $\Sigma^i j_f^n$ admits a homotopy section σ which satisfies $\sigma \Sigma^i f \simeq \Sigma^i \iota_n$.

In order to give the proof of next theorem we will use the following well-known result:

Lemma 15. Let $Y \xrightarrow{f} X \xrightarrow{\lambda} C_f$ be a homotopy cofibre sequence. If $f : Y \rightarrow X$ admits a homotopy retraction r , then there exists a map $\sigma : \Sigma C_f \rightarrow \Sigma X$ such that $\Sigma \lambda \sigma \simeq id$ and $\sigma \Sigma \lambda + \Sigma f \Sigma r \simeq id$.

Theorem 16. If $f : Y \rightarrow X$ admits a homotopy retraction then, for any $i \geq 1$, $\sigma^i \mathrm{secat}(f) = \sigma^i \mathrm{relcat}(f)$.

Proof. Let $i \geq 1$. We just have to prove the inequality $\sigma^i \mathrm{secat}(f) \geq \sigma^i \mathrm{relcat}(f)$. Suppose that $\sigma^i \mathrm{secat}(f) \leq n$ and consider the following homotopy commutative diagram:

$$\begin{array}{ccccc} & & \Sigma^i *_X^n Y & & \\ & \nearrow \Sigma^i \iota_n & \downarrow \Sigma^i j_f^n & & \\ \Sigma^i Y & \xrightarrow{\Sigma^i f} & \Sigma^i X & \xrightarrow{\Sigma^i \lambda} & \Sigma^i C_f \end{array}$$

By Lemma 15 we know that there exists a map $\sigma : \Sigma^i C_f \rightarrow \Sigma^i X$ such that $\Sigma^i \lambda \sigma \simeq id$ and $\sigma \Sigma^i \lambda + \Sigma^i f \Sigma^i r \simeq id$. Let $s : \Sigma^i X \rightarrow \Sigma^i *_X^n Y$ be the homotopy section of $\Sigma^i j_f^n$ given by the hypothesis $\sigma^i \text{secat}(f) \leq n$ and set $s' := s \sigma \Sigma^i \lambda + \Sigma^i \iota_n \Sigma^i r$. We then have:

$$\Sigma^i j_f^n s' = \Sigma^i j_f^n s \sigma \Sigma^i \lambda + \Sigma^i j_f^n \Sigma^i \iota_n \Sigma^i r = \sigma \Sigma^i \lambda + \Sigma^i f \Sigma^i r = id.$$

Therefore s' is a homotopy section of $\Sigma^i j_f^n$. In addition, since $\Sigma^i f$ is a co-H-map and $\lambda f \simeq *$, we have

$$s' \Sigma^i f \simeq s \sigma \Sigma^i \lambda \Sigma^i f + \Sigma^i \iota_n \Sigma^i r \Sigma^i f \simeq \Sigma^i \iota_n.$$

This means that $\sigma^i \text{relcat}(f) \leq n$. □

If X is a topological space, then we can straightforwardly define

$$\sigma^i \text{TC}(X) := \sigma^i \text{secat}(\Delta); \quad \sigma^i \text{TC}^M(X) := \sigma^i \text{relcat}(\Delta)$$

Corollary 17. Let X be a space. For $i \geq 1$ one has $\sigma^i \text{TC}(X) = \sigma^i \text{TC}^M(X)$.

4 A Bernstein-Hilton weak version of the D-EH conjecture.

Here we will consider weak versions of sectional and relative categories in the sense of Bernstein-Hilton and prove that the corresponding D-EH conjecture for these invariants holds. Recall that the relative category of a map $f : Y \rightarrow X$ has a Whitehead characterization [4]. Indeed, for each n we can consider the n -th fat wedge construction

$$t_n : T^n(f) \rightarrow X^{n+1}$$

inductively defined as follows. For $n = 0$ we have $T^0(f) = Y$ and $t_0 = f : Y \rightarrow X$. If $t_{n-1} : T^{n-1}(f) \rightarrow X^n$ is defined, then t^n is the join map

$$\begin{array}{ccc} \bullet & \xrightarrow{\quad} & X^n \times Y \\ \downarrow & \nearrow & \downarrow id_{X^n} \times f \\ & T^n(f) & \\ \downarrow & \nearrow & \downarrow \\ T^{n-1}(f) \times X & \xrightarrow{t_{n-1} \times id_X} & X^{n+1}. \end{array}$$

We know that there exists a homotopy pullback (see [4, Th. 25] or [9, Th. 8])

$$\begin{array}{ccc} *_X^n Y & \xrightarrow{j_f^n} & X \\ \varepsilon_n \downarrow & & \downarrow \Delta_{n+1} \\ T^n(f) & \xrightarrow{t_n} & X^{n+1}. \end{array}$$

Then we can also consider the following homotopy commutative square, where τ_n is the composite $Y \rightarrow *_X^n Y \xrightarrow{\varepsilon_n} T^n(f)$:

$$\begin{array}{ccc} Y & \xrightarrow{\tau_n} & T^n(f) \\ f \downarrow & & \downarrow t_n \\ X & \xrightarrow{\Delta_{n+1}} & X^{n+1}. \end{array}$$

Proposition 18. [4, Prop. 26] Let $f : Y \rightarrow X$ be an arbitrary map. Then $\text{relcat}(f) \leq n$ if and only if there exists a map $\varphi : X \rightarrow T^n(f)$ making commutative, up to homotopies, the following diagram

$$\begin{array}{ccc} Y & \xrightarrow{\tau_n} & T^n(f) \\ f \downarrow & \nearrow \varphi & \downarrow t_n \\ X & \xrightarrow{\Delta_{n+1}} & X^{n+1}. \end{array}$$

In order to get a more manageable description of $T^n(f)$ and for the sake of simplicity we will suppose in this section that $f : Y \rightarrow X$ is a cofibration and we may therefore consider the identification $Y \equiv f(Y)$. Observe that by a cofibration we mean a *closed* map having the usual homotopy extension property.

Proposition 19. [9, Cor. 11] Let $f : Y \hookrightarrow X$ be a cofibration. Then the n -th sectional fat wedge $t_n : T^n(f) \hookrightarrow X^{n+1}$ is, up to homotopy equivalence,

$$T^n(f) = \{(x_0, x_1, \dots, x_n) \in X^{n+1} : x_i \in Y \text{ for some } i\},$$

t_n being the natural inclusion. Moreover, t_n is a cofibration.

In this case one can check that $\tau_n : Y \rightarrow T^n(f)$ is given, up to homotopy equivalence, as $\tau_n(a) = (a, a, \dots, a)$. If $\Delta_{n+1} : X \rightarrow X^{n+1}$ denotes the diagonal map, then there is a strictly commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\tau_n} & T^n(f) \\ f \downarrow & & \downarrow t_n \\ X & \xrightarrow{\Delta_{n+1}} & X^{n+1}. \end{array}$$

Now we introduce a refined version of Proposition 18. In order to do this we need the following well-known result whose proof can be found for instance in [17].

Lemma 20. Suppose $f : Y \hookrightarrow X$ a cofibration and $\varphi : X \rightarrow X$ a map such that $\varphi f = f$ and $\varphi \simeq id_X$. Then there exists a map $\psi : X \rightarrow X$ such that $\psi f = f$ and $\psi \varphi \simeq id_X \text{ rel } Y$.

Proposition 21. Let $f : Y \hookrightarrow X$ be a cofibration. Then $\text{relcat}(f) \leq n$ if and only if there exists a map $\phi : X \rightarrow T^n(f)$ such that $\phi f = \tau_n$ and $t_n \phi \simeq \Delta_{n+1} \text{ rel } Y$.

Proof. Suppose that $\text{relcat}(f) \leq n$ and take $\varphi : X \rightarrow T^n(f)$ such that $\varphi f \simeq \tau_n$ and $t_n \varphi \simeq \Delta_{n+1}$. Since f is a cofibration we can suppose without loss of generality that $\varphi f = \tau_n$ and $t_n \varphi \simeq \Delta_{n+1}$. Take a homotopy $L : t_n \varphi \simeq \Delta_{n+1}$ and consider the notation $t_n \varphi = (\varphi_0, \dots, \varphi_n)$, $L = (L_0, \dots, L_n)$ with $\varphi_i : X \rightarrow X$ and $L_i : X \times I \rightarrow X$ for all $i \in \{0, 1, \dots, n\}$. Note that $\varphi_i f = f$ and that $L_i : \varphi_i \simeq id_X$. Therefore, by previous lemma, we can find a map $\psi_i : X \rightarrow X$ such that $\psi_i f = f$ and a homotopy $L'_i : \psi_i \varphi_i \simeq id_X \text{ rel } A$. We set

$$\phi_i := \psi_i \varphi_i : X \rightarrow X.$$

Taking into account that $(\phi_0(x), \dots, \phi_n(x)) \in T^n(f)$ for all $x \in X$, we obtain a map $\phi : X \rightarrow T^n(f)$ such that $t_n \phi = (\phi_0, \dots, \phi_n)$. Obviously, $\phi f = \tau_n$ and $L' = (L'_0, \dots, L'_n)$ is a homotopy $L' : t_n \phi \simeq \Delta_{n+1} \text{ rel } Y$. \square

If $f : Y \hookrightarrow X$ is a cofibration then, for each $n \geq 0$ we can take the cofibre sequence $T^n(f) \xrightarrow{t_n} X^{n+1} \xrightarrow{q_n} X^{n+1}/T^n(f)$ obtaining a diagram

$$\begin{array}{ccc} & & T^n(f) \\ & & \downarrow t_n \\ X & \xrightarrow{\Delta_{n+1}} & X^{n+1} \\ & & \downarrow q_n \\ & & X^{n+1}/T^n(f). \end{array}$$

Recall from [9] that the *weak sectional category* of f , $\text{wsecat}(f)$, is defined as the least n such that $q_n \Delta_{n+1} \simeq *$.

Definition 22. We define the *weak relative category* of $f : Y \hookrightarrow X$, denoted $\text{wrelcat}(f)$, as the least n such that $q_n \Delta_{n+1} \simeq * \text{ rel } Y$.

Proposition 23. Let $f : Y \hookrightarrow X$ be a cofibration. Then the following chain of inequalities holds

$$\text{nil } H^*(X, Y) \leq \text{wcat}(X/Y) \leq \text{wrelcat}(f) \leq \text{relcat}(f).$$

Proof. From the pushout

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & * \\ f \downarrow & & \downarrow \\ X & \xrightarrow{p} & X/Y \end{array}$$

we obtain the following strictly commutative diagram, where the top square is a homotopy pushout (see [9, Prop. 12]) and the bottom square is induced by the homotopy cofibre construction where the induced map w is a homotopy equivalence:

$$\begin{array}{ccc}
T^n(f) & \xrightarrow{\quad} & T^n(X/Y) \\
\downarrow t_n & & \downarrow j_n \\
X \xrightarrow{\Delta_{n+1}} X^{n+1} & \xrightarrow{p^{n+1}} & (X/Y)^{n+1} \\
\downarrow q_n & & \downarrow q'_n \\
X^{n+1}/T^n(f) & \xrightarrow[\simeq]{w} & (X/Y)^{[n+1]}.
\end{array}$$

Now, if $\text{wrelcat}(f) = n$ and we take a homotopy $H : X \times I \rightarrow X^{n+1}/T^n(f)$ with $H : q_n \Delta_{n+1} \simeq * \text{ rel } Y$, then we can define

$$\bar{H} : X/Y \times I \rightarrow (X/Y)^{[n+1]}$$

by $\bar{H}([x], t) := wH(x, t)$. Then \bar{H} is a well defined continuous map such that $\bar{H} : q'_n \Delta_{n+1} \simeq *$. This proves that $\text{wcat}(X/Y) \leq \text{wrelcat}(f)$. Therefore

$$\text{nil } H^*(X, Y) = \text{cuplength}(X/Y) \leq \text{wcat}(X/Y) \leq \text{wrelcat}(f).$$

On the other hand, if $\text{relcat}(f) = n$, then by Proposition 21 there exists a map $\phi : X \rightarrow T^n(f)$ such that $\phi f = \tau_n$ and $t_n \phi \simeq \Delta_{n+1} \text{ rel } Y$. Therefore

$$q_n \Delta_{n+1} \simeq q_n t_n \phi = * \text{ rel } Y$$

and $\text{wrelcat}(f) \leq n$. □

Remark 24. If f^* denotes the induced homomorphism in cohomology, then using Theorem 21(d) of [9] we immediately have the following chain of inequalities

$$\text{nil ker}(f^*) \leq \text{wsecat}(f) \leq \text{wrelcat}(f) \leq \text{relcat}(f).$$

It is natural to ask whether $\text{nil ker}(f^*)$ and $\text{nil } H^*(X, Y)$ are related or not. In Theorem 21(e) of [9] it was actually established that, if f has a homotopy retraction, then $\text{wsecat}(f) = \text{wcat}(X/Y)$ and

$$\text{nil ker}(f^*) = \text{cuplength}(X/Y) = \text{nil } H^*(X, Y).$$

And finally, our last result in this section

Theorem 25. Let $f : Y \hookrightarrow X$ be a cofibration. Then $\text{wrelcat}(f) = \text{wcat}(X/Y)$ holds. In particular, if f admits a homotopy retraction, then $\text{wrelcat}(f) = \text{wsecat}(f)$.

Proof. It only remains to prove that $\text{wrelcat}(f) \leq \text{wcat}(X/Y)$. So suppose that $\text{wcat}(X/Y) = n$ and take a homotopy $H : X/Y \times I \rightarrow (X/Y)^{[n+1]}$ such that $H : \Delta_{n+1}q'_n \simeq *$. Then the composite

$$X \times I \xrightarrow{p \times \text{id}} X/Y \times I \xrightarrow{H} (X/Y)^{[n+1]}$$

clearly gives a homotopy $q'_n \Delta_{n+1} p \simeq * \text{ rel } Y$. But from the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{p} & X/Y \\ \Delta_{n+1} \downarrow & & \downarrow \Delta_{n+1} \\ X^{n+1} & \xrightarrow{p^{n+1}} & (X/Y)^{n+1} \\ q_n \downarrow & & \downarrow q'_n \\ X^{n+1}/T^n(f) & \xrightarrow[\simeq]{w} & (X/Y)^{[n+1]} \end{array}$$

we have that $q'_n \Delta_{n+1} p = w q_n \Delta_{n+1}$ and therefore $w q_n \Delta_{n+1} \simeq * \text{ rel } Y$.

Now take the commutative square of solids arrows

$$\begin{array}{ccc} Y & \xrightarrow{*} & X^{n+1}/T^n(f) \\ f \downarrow & \nearrow & \downarrow \simeq w \\ X & \xrightarrow[*]{} & (X/Y)^{[n+1]}. \end{array}$$

As $q_n \Delta_{n+1} : X \rightarrow X^{n+1}/T^n(f)$ and the constant map $* : X \rightarrow X^{n+1}/T^n(f)$ are two liftings of this square, by the *Lifting Lemma* [1, page 90] we have that

$$q_n \Delta_{n+1} \simeq * \text{ rel } Y,$$

but this means that $\text{wrelcat}(f) \leq n$.

The second part of the theorem follows from the fact that $\text{wsecat}(f) = \text{wcat}(X/Y)$ when f admits a homotopy retraction (see [9]). \square

Recall that for a locally equiconnected space X the diagonal map $\Delta : X \rightarrow X \times X$ is a cofibration. Therefore we naturally set

$$\text{wTC}^M(X) := \text{wrelcat}(\Delta)$$

and we directly have the following corollary. Observe that, by definition in [9], $\text{wTC}(X) = \text{wsecat}(\Delta)$.

Corollary 26. If X is a locally equiconnected space, then

$$\text{wTC}(X) = \text{wTC}^M(X) = \text{wcat}(X \times X/\Delta(X)).$$

Dranishnikov conjectured in [6] that $\mathrm{TC}^M(X) = \mathrm{cat}(X \times X / \Delta(X))$. The second equality of the above corollary can be seen as a positive answer to a weak version of this conjecture.

Remark 27. From Iwase-Sakai's characterization of $\mathrm{TC}^M(X)$ in the pointed fibrewise setting (see [13], [12]) A. Franc and P. Pavešić introduced in [8] some lower bounds for $\mathrm{TC}^M(X)$. In particular they defined stable and weak versions of $\mathrm{TC}^M(X)$. It is possible to check that these invariants coming from the pointed fibrewise setting are upper bounds for our $\sigma^i \mathrm{TC}^M(X)$ and $\mathrm{wTC}^M(X)$ respectively. However we do not know whether they are the same.

5 The D-EH conjecture in rational homotopy theory.

In this section we assume that $f : Y \rightarrow X$ is a map between simply-connected spaces of finite type over \mathbb{Q} and we consider the rationalization $f_0 : Y_0 \rightarrow X_0$. In this context the D-EH conjecture reads as: if $f : Y \rightarrow X$ admits a homotopy retraction then $\mathrm{secat}(f_0) = \mathrm{relcat}(f_0)$.

The sectional category of f_0 can be characterized as follows in terms of any surjective model of f in the category **cdga** of commutative differential graded algebras:

Proposition 28. [3] Let $f : Y \rightarrow X$ be a map with surjective **cdga** model $\varphi : (A, d) \rightarrow (B, d)$ and let $K = \mathrm{Ker} \varphi$. Then $\mathrm{secat}(f_0)$ is the smallest n for which there exists a **cdga** morphism τ such that $\tau \circ i = \mu_{n+1}$,

$$\begin{array}{ccc} A^{\otimes n+1} & \xrightarrow{\mu_{n+1}} & A \\ \downarrow i & \nearrow \tau & \\ (A^{\otimes n+1} \otimes \Lambda W_{n+1}, D) & & \end{array}$$

where i is a relative Sullivan model for the projection $\pi : A^{\otimes m+1} \rightarrow A^{\otimes m+1} / K^{\otimes n+1}$.

In order to estimate in terms of this data the relative category of f_0 we consider the map $k_n : A \rightarrow (A \otimes \Lambda W_{n+1}, \overline{D})$ given by the pushout of μ_{m+1} and i . It is easy to see that the existence of the map τ in previous proposition is equivalent to the existence of a homotopy retraction for k_n . In fact, k_n is a model for the join map $j_f^n : *_f^n Y \rightarrow X$ and the morphism l_n induced in following diagram is a model for the map ι_n in Diagram (1).

$$\begin{array}{ccccc}
& & \frac{A^{\otimes n+1}}{K^{\otimes n+1}} & & (2) \\
& \nearrow \pi & \uparrow \theta \simeq & \searrow \overline{\Pi_{n+1} \varphi} & \\
A^{\otimes n+1} & \xrightarrow{i} & (A^{\otimes n+1} \otimes \Lambda W_{n+1}, D) & & \\
\downarrow \mu_{n+1} & & \downarrow & & \\
A & \xrightarrow{k_n} & (A \otimes \Lambda W_{n+1}, \bar{D}) & & \\
& \searrow \varphi & \swarrow l_n & \searrow & \\
& & & & B
\end{array}$$

We can choose a relative model i for π such that the quasi-isomorphism θ satisfies $\theta(W_{n+1}) = 0$. In this case, we have, for any $\omega \in W_{n+1}$, $D\omega \in K^{\otimes n+1} \oplus A^{\otimes n+1} \otimes \Lambda^+ W_{n+1}$ and $\bar{D}\omega \in K \oplus A \otimes \Lambda^+ W_{n+1}$. Furthermore the induced morphism l_n is such that $l_n(a) = \varphi(a)$ if $a \in A$ and $l_n(W_{n+1}) = 0$. These remarks lead to

Proposition 29. Let $f: Y \rightarrow X$ be a map and $\varphi: A \rightarrow B$ a surjective **cdga** model for f with $K = \text{Ker } \varphi$. If there exists a **cdga** morphism $\tau: (A \otimes \Lambda W_{n+1}, \bar{D}) \rightarrow A$ such that $\tau \circ k_n = \text{Id}_A$ and $\tau(W_{n+1}) \subset K$ then $\text{relcat}(f_0) \leq n$.

Consider now the quotient map $p_n: (A, d) \rightarrow (A/K^{n+1}, \bar{d})$. Let

$$\bar{\varphi}: (A/K^{n+1}, \bar{d}) \rightarrow (B, d)$$

be the morphism induced by φ . The following commutative diagram

$$\begin{array}{ccccc}
A^{\otimes n+1} & \xrightarrow{\pi} & \frac{A^{\otimes n+1}}{K^{\otimes n+1}} & & \\
\downarrow \mu_{n+1} & & \downarrow & & \searrow \overline{\Pi_{n+1} \varphi} \\
A & \xrightarrow{\quad} & A/K^{n+1} & & \\
& \searrow \varphi & \searrow \bar{\varphi} & \searrow & \\
& & & & B,
\end{array}$$

where the second vertical morphism is induced by the multiplication, permits us to see that the morphism l_n of Diagram 2 factors as

$$A \otimes \Lambda W_{n+1} \xrightarrow{\lambda_n} A/K^{n+1} \xrightarrow{\bar{\varphi}} B.$$

The morphism λ_n satisfies $\lambda_n k_n = p_n$ and $\lambda_n(W_{n+1}) \subset \text{Ker}(\bar{\varphi})$. Observe now that, if $K^{n+1} = 0$, then $p_n = \text{Id}_A$ and $\bar{\varphi} = \varphi$. Therefore the morphism $\tau := \lambda_n$

satisfies the conditions that give $\text{relcat}(f_0) \leq n$ in Proposition 29. We hence obtain the following result where $\text{nil}(K)$ denotes the maximal length of a non trivial product in K .

Corollary 30. Let $f: Y \rightarrow X$ be a map and $\varphi: A \rightarrow B$ a surjective **cdga** model for f with $K = \text{Ker } \varphi$. Then $\text{relcat}(f_0) \leq \text{nil}(K)$.

We now specialize this discussion in the case of $f = \Delta: X \rightarrow X \times X$. Since $\text{relcat}(\Delta) = \text{TC}^M(X)$ we write $\text{TC}_0^M(X)$ instead of $\text{relcat}(\Delta_0)$. A surjective **cdga** model of Δ is given by the multiplication $\mu_A: A \otimes A \rightarrow A$ where (A, d) is any **cdga** model of X . We thus obtain:

Corollary 31. Let X be a space and let (A, d) be a **cdga** model of X . Then

$$\text{TC}_0(X) \leq \text{TC}_0^M(X) \leq \text{nil}(\ker \mu_A).$$

In particular, if X admits a **cdga** model (A, d) such that $\text{TC}_0(X) = \text{nil}(\ker \mu_A)$, then $\text{TC}_0(X) = \text{TC}_0^M(X)$.

Using this result together with Theorem 1.4 and Corollary 2.2 of [15] we can exhibit two important classes of spaces for which the rational version of the Iwase-Sakai conjecture is true:

Corollary 32. Let X be a simply-connected space. If X is formal or has its rational homotopy, $\pi_*(X) \otimes \mathbb{Q}$, of finite dimension and concentrated in odd degrees, then $\text{TC}_0(X) = \text{TC}_0^M(X)$.

We finish this section with a weak version of the D-EH conjecture that we can establish in the framework of rational homotopy theory. As in the D-EH conjecture, we suppose that the map $f: Y \rightarrow X$ admits a homotopy retraction.

Using standard techniques, we can consider a **cdga** model $\varphi: (A, d) \rightarrow (B, d)$ of f which admits a strict section, *i.e.* there exists a **cdga** morphism $s: B \rightarrow A$ such that $\varphi s = \text{Id}_B$. Considering Diagram (2), the morphism s makes k_n a (B, d) -module morphism and we define:

- $\text{m}_{\text{Bsecat}}(\varphi)$ as the smallest n such that k_n admits a (B, d) -module retraction r ;
- $\text{m}_{\text{Brelcat}}(\varphi)$ as the smallest n such that k_n admits a (B, d) -module retraction r with $r(\Lambda^+ W_{n+1}) \subset K$.

Our definition of $\text{m}_{\text{Brelcat}}(\varphi)$ provides actually an upper bound of a (B, d) -module version of the relative category in the strict sense but it is not necessary to introduce an intermediate notion since we have:

Proposition 33. $\text{m}_{\text{Brelcat}}(\varphi) = \text{m}_{\text{Bsecat}}(\varphi)$.

Proof. We just have to prove that $m_{\text{Brelcat}}(\varphi) \leq m_{\text{Bsecat}}(\varphi)$. Suppose there is a (B, d) -module morphism $r: (A \otimes \Lambda W_{n+1}, \overline{D}) \rightarrow A$ such that $r(a) = a$ for all $a \in A$. Define $r': (A \otimes \Lambda W_{n+1}, \overline{D}) \rightarrow A$ as $r'(a) := a$ and $r'(a\omega) := r(a\omega) - s\varphi r(a\omega)$ for $\omega \in \Lambda^+ W_{n+1}$. It is obvious that $r'(\Lambda^+ W_{n+1}) \subset K$ and that $r'(s(b)a\omega) = s(b)r'(a\omega)$. We shall now see that r' commutes with differentials. Write $\overline{D}\omega = \alpha + \sum_i a_i \psi_i$, with $\alpha \in K$, $\{a_i\}_i \subset A$ and $\{\psi_i\}_i \subset \Lambda^+ W$. Since $\alpha \in K$ we have $\varphi r(a\alpha) = 0$. Therefore

$$\begin{aligned} r'(\overline{D}(a\omega)) &= r'((da)\omega) + (-1)^{|a|} r'(a(\overline{D}\omega)) = (r((da)\omega) - s\varphi r((da)\omega) + \\ &(-1)^{|a|} \left(r(a\alpha) + r\left(a \sum_i a_i \psi_i\right) - s\varphi r(a\alpha) - s\varphi r\left(a \sum_i a_i \psi_i\right) \right) = \\ &r(\overline{D}(a\omega)) - s\varphi r(\overline{D}(a\omega)) = \overline{D}(r'(a\omega)). \end{aligned}$$

This implies that $m_{\text{Brelcat}}(\varphi) \leq n$. □

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